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Relations between different approaches to the relativistic two-body problem

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Abstract. The relativistic and quantum mechanical two-body problem with interaction has been a controversial matter for many decades. In this paper we show that two very different approaches to the problem for a particular type of interaction lead to the same final equation. Furthermore, the second of them can be solved exactly in an elementary fashion, and leads to an equally spaced spectrum of a familiar type for the square of the total energy.

1. Introduction

In recent years the two-body relativistic problem has been visualized from many different angles. In this paper we shall consider only two of the approaches. The first one by Moshinsky *et al* [1,2] emphasizes a *single* relativistic equation for the two-body problem with an interaction which is suggested by the approach for two free Dirac particles, as well as by a variational analysis of Barut on the Lagrangian of quantum electrodynamics [3]. The second, in which we mainly emphasize the work of Sazdjian [4] and Crater and van Alstine [5], starts from *two* independent Dirac equations where an interaction is introduced in a way in which they remain compatible.

We shall compare these two approaches, emphasizing an interaction which we called the Dirac oscillator [6], as it is a particularly simple one, though our analysis allows much more general interactions. This will be stressed later on in this article.

The concept of the Dirac oscillator for the one-particle system was introduced several years ago [6] by replacing the momentum p in the Dirac equation by

$$p \rightarrow p - \mathrm{i}\omega r\beta \tag{1}$$

where we used units in which $\hbar = m = c = 1$ and β is the matrix referred to usually as the Dirac matrix [7].

2. A single relativistic equation for two particles

Almost immediately the problem of one body was generalized to two bodies interacting through a Dirac oscillator, using a single Poincaré invariant equation of the form [1,2]

$$\sum_{s=1}^{2} \left\{ \Gamma_s[\gamma_s^\mu (p_{\mu s} - \mathrm{i}\omega x'_{\mu s}\Gamma) + 1] \right\} \psi = 0 \tag{2}$$

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where repeated indices μ indicate summation over $\mu = 0, 1, 2, 3$, while $p_{\mu s}$, $s = 1, 2$ is the four-momentum of particle s and γ_s^μ the corresponding γ matrix, while

$$x'_{\mu s} = x_{\mu s} - \tfrac{1}{2}(x_{\mu 1} + x_{\mu 2}) \tag{3}$$

and

$$\Gamma = (\gamma_1^\mu u_\mu)(\gamma_2^\mu u_\mu) \qquad \Gamma_1 = (\gamma_2^\mu u_\mu) \qquad \Gamma_2 = (\gamma_1^\mu u_\mu) \tag{4}$$

where u_μ is a unit timelike four-vector given in terms of the total momentum P^μ by

$$u_\mu = P_\mu(-P_\tau P^\tau)^{-1/2}. \tag{5}$$

We assume also that the two particles have equal mass $m_1 = m_2 = 1$.

In the centre-of-mass frame when $P_i = 0$, $i = 1, 2, 3$, and where $P^0 = -P_0$ is the total energy, which will be denoted by E, equation (2) becomes [1,2]

$$\left[(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2)\cdot\left(\boldsymbol{p} - \mathrm{i}\frac{\omega}{2}\boldsymbol{r}\beta_1\beta_2\right) + (\beta_1 + \beta_2)\right]\psi = E\psi \tag{6}$$

where $\boldsymbol{r}$, $\boldsymbol{p}$ are the relative coordinate and momentum, respectively, of the two particles, and α_s, β_s, $s = 1, 2$ are the Dirac matrices for the two particles in the Dirac equation [7].

The energy spectrum E, as well as the eigenstates, were derived explicitly in an elementary fashion [1,2].

We note at this point that equation (2) is not the only single Poincaré invariant equation that can be derived for the two-body problem. For example, if we consider the matrices usually denoted by

$$\gamma_{5s} = \mathrm{i}\gamma_s^0\gamma_s^1\gamma_s^2\gamma_s^3 \qquad s = 1, 2 \tag{7}$$

they are pseudoscalar ones, but $\gamma_{51}\gamma_{52}$ would be a scalar, so we can replace Γ in (2) by

$$\Gamma\gamma_{51}\gamma_{52} \tag{8}$$

which would not alter the Poincaré invariant, while in the centre-of-mass frame we would have to replace $\beta_1\beta_2$ in equation (6) by $\beta_1\beta_2\gamma_{51}\gamma_{52}$, leaving the equation exactly solvable, as will be shown below.

Having presented the formulation of Moshinsky *et al* [1,2] for the two-body problem with a Dirac oscillator interaction, through a single Poincaré invariant equation, we would like to turn now to a more familiar analysis of this problem using two compatible single-particle equations as carried out by many authors, in particular by Sazdjian [4] and Crater and van Alstine [5].

3. A system of two relativistic equations

We shall consider the single free-particle Dirac equations in the form used by Crater and van Alstine [5], i.e.

$$\gamma_{5s}(\gamma_s\cdot p_s + 1)\psi = 0 \qquad s = 1, 2 \tag{9}$$

where we replace the repeated index $\mu = 0, 1, 2, 3$ by a dot; when passing to the system with an interaction we follow the notation of Sazdjian [4], i.e.

$$\{\gamma_{51}(\gamma_1\cdot p_1 + 1) + V_1\}\psi = 0 \tag{10a}$$

$$\{\gamma_{52}(\gamma_2\cdot p_2 + 1) + V_2\}\psi = 0 \tag{10b}$$

and the first problem is to prove the compatibility of this system. This is achieved [4,5] by multiplying the first equation by $\gamma_{51}(\gamma_1\cdot p_1+1)$ and the second by $\gamma_{52}(\gamma_2\cdot p_2+1)$ and substracting, so that from the properties of the γ matrices we get [4]

$$\left\{p_1^2-p_2^2+\gamma_{51}(\gamma_1\cdot p_1+1)V_1-\gamma_{52}(\gamma_2\cdot p_2+1)V_2\right\}\psi=0. \tag{11}$$

Now as the masses of the two particles are equal and, in our units, take the value 1, we can write

$$p_1=p+(P/2) \qquad p_2=-p+(P/2) \tag{12}$$

where P is the total four-momentum while p is the relative one, and, for simplicity, we suppressed the index μ.

From (12) we see that

$$(p_1^2-p_2^2)=2P\cdot p. \tag{13}$$

It is usual to require [4,5] that this term, when applied to ψ, should vanish because then in the centre-of-mass frame (i.e. when $P_i=0, i=1,2,3$) it becomes $2P^0p_0$ and since P^0 (i.e. energy E) does not vanish, we expect that the application of $p_0=-\mathrm{i}\partial/\partial x^0$ to ψ vanishes; hence, ψ is independent of the relative time x^0, a fundamental requirement for the equations to have physical sense [4,5]. Thus if $P\cdot p\psi=0$, we see from (11) that we require that

$$\gamma_{51}(\gamma_1\cdot p_1+1)V_1\psi=\gamma_{52}(\gamma_2\cdot p_2+1)V_2\psi \tag{14}$$

which implies that

$$V_1=\gamma_{52}(\gamma_2\cdot p_2+1)V \qquad V_2=\gamma_{51}(\gamma_1\cdot p_1+1)V \tag{15}$$

where V has to be a Poincaré invariant so that equations (10) also have this property. The simplest way to achieve this behaviour is to define the transverse relative coordinate [4,5]

$$x_\perp^{\mu'}=x^{\mu'}-(P_\sigma x^{\sigma'})P^\mu(P_\tau P^\tau)^{-1} \tag{16a}$$

$$x^{\mu'}=x_1^\mu-x_2^\mu \tag{16b}$$

and to define the scalar formed from its contraction with itself as

$$\rho^2=x_\perp^{\mu'}x'_{\mu\perp} \tag{17}$$

so we shall assume that V is a function of ρ only.

We now have a set of two compatible equations

$$\left\{\gamma_{51}(\gamma_1\cdot p_1+1)+\gamma_{52}(\gamma_2\cdot p_2+1)V\right\}\psi=0 \tag{18a}$$

$$\left\{\gamma_{52}(\gamma_2\cdot p_2+1)+\gamma_{51}(\gamma_1\cdot p_1+1)V\right\}\psi=0. \tag{18b}$$

If in (18a) we transfer V to the left of the operator $\gamma_{52}(\gamma_2\cdot p_2+1)$, adding a term from the commutation which gives a derivative of V with respect to x_2^μ, and then use (18b) to substitute the term $V\gamma_{52}(\gamma_2\cdot p_2+1)$ appearing in (18a), we get

$$\left\{\gamma_{51}(\gamma_1\cdot p_1+1)-V\gamma_{51}(\gamma_1\cdot p_1+1)V-\mathrm{i}\gamma_{52}(\gamma_2\cdot\partial_2V)\right\}\psi=0 \tag{19}$$

where $\gamma_2\cdot\partial_2=\gamma_2^\mu\partial/\partial x_2^\mu$.

Moving all the terms V in (19) to the left-hand side and multiplying the resulting equation by $\gamma_{51}(1-V^2)^{-1}$, we finally get

$$\left\{(\gamma_1\cdot p_1+1)+(1-V^2)^{-1}\left[-\mathrm{i}\gamma_{51}\gamma_{52}(\gamma_2\cdot\partial_2V)+\mathrm{i}V(\gamma_1\cdot\partial_1V)\right]\right\}\psi=0. \tag{20a}$$

Carrying out a similar analysis, but starting from (18*b*), we obtain in turn

$$\left\{(\gamma_2 \cdot p_2 + 1) + (1 - V^2)^{-1}\left[-\mathrm{i}\gamma_{51}\gamma_{52}(\gamma_1 \cdot \partial_1 V) + \mathrm{i}V(\gamma_2 \cdot \partial_2 V)\right]\right\}\psi = 0. \tag{20b}$$

We now make a change in our function ψ using the relation

$$\psi = (1 - V^2)^{-1/2}\phi \tag{21}$$

and consider the replacements of p_1, p_2 indicated in (12), using also the fact that as V is a function of ρ, so

$$(\partial V/\partial x_s^\mu) = (\partial V/\partial \rho)(\partial \rho/\partial x_s^\mu) \equiv \dot{V}(x'_{\mu\perp}/\rho)(-1)^{s+1}. \tag{22}$$

We finally obtain that the equations (20) reduce to

$$\left\{(\gamma_1 \cdot p + \tfrac{1}{2}\gamma_1 \cdot P + 1) + (1 - V^2)^{-1}\dot{V}\left[\mathrm{i}\gamma_{51}\gamma_{52}(\gamma_2 \cdot x'_\perp)\rho^{-1}\right]\right\}\phi = 0 \tag{23a}$$

$$\left\{(-\gamma_2 \cdot p + \tfrac{1}{2}\gamma_2 \cdot P + 1) + (1 - V^2)^{-1}\dot{V}\left[-\mathrm{i}\gamma_{51}\gamma_{52}(\gamma_1 \cdot x'_\perp)\rho^{-1}\right]\right\}\phi = 0. \tag{23b}$$

Turning now to the centre-of-mass frame where $\gamma_1 \cdot P = \beta_1 P_0 = -\beta_1 E$, $\gamma_2 \cdot P = \beta_2 P_0 = -\beta_2 E$ and multiplying (23*a*) by β_1, and (23*b*) by β_2, recalling that ψ is independent of x_0, and using the relations for the three component vectors,

$$\boldsymbol{\gamma}_1 = \beta_1 \boldsymbol{\alpha}_1 \qquad \boldsymbol{\gamma}_2 = \beta_2 \boldsymbol{\alpha}_2 \tag{24}$$

we see that by adding the two equations (23) modified in this fashion we finally obtain

$$\left\{(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2) \cdot \boldsymbol{p} + \beta_1 + \beta_2 + \dot{V}\left[r(1 - V^2)\right]^{-1} \times \left[\mathrm{i}\beta_1\gamma_{51}\gamma_{52}(\boldsymbol{\gamma}_2 \cdot \boldsymbol{r}) - \mathrm{i}\beta_2\gamma_{51}\gamma_{52}(\boldsymbol{\gamma}_1 \cdot \boldsymbol{r})\right]\right\}\phi = E\phi \tag{25}$$

where $\boldsymbol{p}$ and $\boldsymbol{r}$ are the relative spatial three-component vectors of momentum and position, while V is a function of the magnitude r and $\dot{V} = (\mathrm{d}V/\mathrm{d}r)$.

It is now a question of selecting the form of the function $V(r)$. Following Sazdjian [4] and Crater and van Alstine [5], we write V as a hyperbolic tangent but, for our future purposes, the argument is taken as $(\omega r^2/4)$, i.e.

$$V(r) = \tanh(\omega r^2/4). \tag{26}$$

We see then that equation (25) becomes

$$\left\{(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2) \cdot \left(\boldsymbol{p} - \mathrm{i}\frac{\omega}{2}\boldsymbol{r}\beta_1\beta_2\gamma_{51}\gamma_{52}\right) + \beta_1 + \beta_2\right\}\phi = E\phi \tag{27}$$

and thus has a form similar to that of (6) but now with the extra term $\gamma_{51}\gamma_{52}$, which, as we indicated above, is a possible variant of the two-body Dirac oscillator.

In fact, the main point of (27) is that it can be solved explicitly, as we shall now proceed to show.

4. Solution of the problem

Let us first note that $\alpha_1, \alpha_2, \beta_1, \beta_2$ can be written as four-dimensional matrices, for example [1, 2]

$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 & 0 & 0 \\ \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1 \\ 0 & 0 & \sigma_1 & 0 \end{pmatrix}. \tag{28}$$

Furthermore, we introduce the notation

$$a_\pm \equiv (\sigma_1 \cdot p) \pm \mathrm{i}\frac{\omega}{2}(\sigma_2 \cdot r) \qquad b_\pm \equiv -(\sigma_2 \cdot p) \pm \mathrm{i}\frac{\omega}{2}(\sigma_1 \cdot r) \tag{29}$$

so that equation (27) becomes [1, 2]

$$\begin{bmatrix} (2-E) & a_- & b_+ & 0 \\ a_+ & -E & 0 & b_- \\ b_- & 0 & -E & a_+ \\ 0 & b_+ & a_- & (-2-E) \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{12} \\ \phi_{22} \end{bmatrix} = 0 \tag{30}$$

where the function ϕ is replaced by a four-component column vector with terms ϕ_{st}, $s, t = 1, 2$ as indicated in (30).

The second and third rows of (30), when applied to ϕ, allow us to express ϕ_{21}, ϕ_{12} in terms of ϕ_{11}, ϕ_{22}, so that substituting them in the equations coming from the first and fourth rows, we obtain a 2×2 matrix operator equation for the two components ϕ_{11} and ϕ_{22}. Introducing then, as in previous work [1, 2], ϕ_+ and ϕ_- by the definitions

$$\begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \phi_{11} \\ \phi_{22} \end{pmatrix} \tag{31}$$

we finally obtain an equation of the form

$$\begin{bmatrix} A - E^2 & 2E \\ 2E & B - E^2 \end{bmatrix} \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} = 0 \tag{32}$$

where

$$A \equiv (a_- - b_+)(a_+ - b_-) = 4\omega\left[S^2 + (S \cdot \eta)(S \cdot \xi) + L \cdot S\right] \tag{33a}$$

$$B \equiv (a_- + b_+)(a_+ + b_-) = 4\omega\left[\hat{N} - (S \cdot \eta)(S \cdot \xi) - L \cdot S\right] \tag{33b}$$

and

$$\eta = \frac{1}{\sqrt{2}}\left(\omega^{1/2} r - \mathrm{i}\omega^{-1/2} p\right) \qquad \xi = \frac{1}{\sqrt{2}}\left(\omega^{1/2} r + \mathrm{i}\omega^{-1/2} p\right) \tag{34}$$

are the creation and annihilation operators for the three-dimensional oscillator of frequency ω, while

$$L = r \times p = -\mathrm{i}(\eta \times \xi) \qquad S = \tfrac{1}{2}(\sigma_1 + \sigma_2) \qquad \hat{N} = \eta \cdot \xi \tag{35}$$

and we made use of the well known commutation relations of all of these operators.

We note now from (33) that the matrix operator in (32) commutes with the number operator $\hat{N}$, the total angular momentum

$$J = L + S \tag{36}$$

and with the parity, the last because it is invariant under the transformation $\eta \to -\eta$, $\xi \to -\xi$.

As in the case in the ordinary Dirac oscillator [1, 2], the eigenfunctions of (32) can then be expressed in terms of oscillator wavefunctions with spin which we can designate by the ket [1, 2]

$$|N(\ell, s)jm\rangle \tag{37}$$

with the symbols characterizing the total number of quanta (N), orbital angular momentum (ℓ), total spin ($s = 0, 1$) and total angular momentum and projection (j, m).

We see then that for definite (N, j) and spin $s = 0$ we have the single state

$$|N(j, 0)jm\rangle \tag{38}$$

whose parity is $(-1)^j$. For $s = 1$ and parity $(-1)^j$ we also have a single state

$$|N(j,1)jm\rangle \tag{39}$$

while for the parity $-(-1)^j$ we have the two states

$$|N(j\pm 1,1)jm\rangle. \tag{40}$$

To get the eigenvalues E of equation (32), we need to calculate the operators A and B appearing in (33) in the basis of (38), (39) or (40).

For the ket in (38), the result is simple, as for spin $s = 0$ all contributions containing $\boldsymbol{S}$ vanish and the matrix operator (32) becomes the numerical one,

$$\begin{bmatrix} -E^2 & 2E \\ 2E & 4\omega N - E^2 \end{bmatrix} \tag{41}$$

which leads to a secular equation whose roots are

$$E^2 = 4 + 4N\omega \tag{42a}$$

$$E^2 = 0. \tag{42b}$$

The vanishing root leads to a phenomenon which the authors have called 'cockroach nest' [8] and is not significant for our present discussion, while the value (42a) for E^2 indicates the typical equally spaced spectrum of the oscillator but now for E^2 rather than E.

For fixed (N, j) and spin 1 but parity $(-1)^j$, we have the single state (39) and from the relation

$$\boldsymbol{L}\cdot\boldsymbol{S} = \tfrac{1}{2}(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2) \tag{43}$$

as well as the matrix element [1]

$$\langle N(j,1)jm|(\boldsymbol{\eta}\cdot\boldsymbol{S})(\boldsymbol{\xi}\cdot\boldsymbol{S})|N(j,1)jm\rangle = N+1 \tag{44}$$

derived in equation (3.21) of reference [1], we see that the matrix operator (32) becomes the numerical one,

$$\begin{bmatrix} 4\omega(N+2) - E^2 & 2E \\ 2E & -E^2 \end{bmatrix}. \tag{45}$$

This matrix leads to a secular equation whose roots are

$$E^2 = 4 + 4\omega(N+2) \tag{46a}$$

$$E^2 = 0 \tag{46b}$$

where we are again only concerned with (46a) and obtain an equally spaced spectrum for E^2, though displaced by two quanta from the one for $s = 0$.

When (N, j) are fixed but the parity is $-(-1)^j$, the matrix element of $\boldsymbol{L}\cdot\boldsymbol{S}$ with respect to the states $|N(j\pm 1,1)jm\rangle$ is trivial if we use (43). On the other hand, the matrix elements

$$\langle N(\ell',1)jm|(\boldsymbol{S}\cdot\boldsymbol{\eta})(\boldsymbol{S}\cdot\boldsymbol{\xi})|N(\ell,1)jm\rangle \tag{47}$$

with $\ell, \ell' = j \pm 1$ can be evaluated straightforwardly using equations (3.24) of reference [1] and their Hermitian conjugate.

Because of the existence of two states $|N(j\pm 1,1)jm\rangle$ instead of one as before, the 2×2 matrix operator in (32) becomes now the 4×4 numerical matrix

$$\begin{bmatrix} \alpha - E^2 & -\delta & 2E & 0 \\ -\delta & \beta - E^2 & 0 & 2E \\ 2E & 0 & \beta - E^2 & \delta \\ 0 & 2E & \delta & \alpha - E^2 \end{bmatrix} \tag{48}$$

where

$$\alpha \equiv 4\omega(2j+1)^{-1} j(N-j+1) \tag{49a}$$

$$\beta \equiv 4\omega(2j+1)^{-1}(j+1)(N+j+2) \tag{49b}$$

$$\delta^2 \equiv \alpha\beta. \tag{49c}$$

The secular equation associated with (48) reduces drastically to the form

$$E^4[E^2 - (\alpha+\beta+4)]^2 = 0 \tag{50}$$

and so, using (49), we have the double roots

$$E^2 = 4 + 4\omega(N+2) \qquad E^2 = 0 \tag{51}$$

and thus get the same spectra as in the case of spin $s = 1$ and parity $(-1)^j$.

The problem has then, in all cases, an equal spacing between the levels of E^2, and presents also an extraordinary accidental degeneracy, which implies the presence of a symmetric Lie algebra that we plan to analyse in another publication.

Before concluding, we must remember that our equation (27) was derived from the sum of equations (23) when written in the centre-of-mass frame, and appropriately modified. What happens when, with the modifications we considered, we look at the difference between these equations, again in the centre-of-mass frame? We note that E no longer appears in the resulting equation and, again following an analysis very similar to the one that led from (27) to (32), we obtain that now ϕ_+, ϕ_- must also obey the equations

$$\{[(\sigma_1 - \sigma_2)\cdot\eta][(\sigma_1 + \sigma_2)\cdot\eta]\}\,\phi_- = 0 \tag{52a}$$

$$\{[(\sigma_1 - \sigma_2)\cdot\xi][(\sigma_1 + \sigma_2)\cdot\xi]\}\,\phi_+ = 0. \tag{52b}$$

It is easy to show though that using the relation

$$\sigma_i\sigma_j = \delta_{ij} + \mathrm{i}\epsilon_{ijk}\sigma_k \tag{53}$$

for both Pauli matrices σ_1 and σ_2, the curly brackets vanish identically and thus they do not impose any further restriction on ϕ_+, ϕ_-, as we could have expected from the compatible character of the two equations (10) required from the beginning.

We note that our spectrum for the square of the energy in all cases can be written as

$$E^2 - 4 = 4\omega[N + s(s+1)] \tag{54}$$

where N is the total number of quanta and s is the spin of the two-particle system $s = 0, 1$.

For the non-relativistic limit we can write

$$E = 2 + \epsilon \tag{55}$$

as the joint mass of the two particles is 2 in our units, and we can consider $\epsilon \ll 1$. Disregarding ϵ^2, we then obtain

$$\epsilon = \omega[N + s(s+1)] \tag{56}$$

which is a very natural expression for the non-relativistic energy of the oscillator.

5. Conclusion

Our final remark is that a very convenient *single* equation for the relativistic two-body problem with a Dirac oscillator interaction can be derived from very different approaches. In one of them [1, 2] we actually start from one single equation for two free Dirac particles, rewrite it in a Poincaré invariant form and replace the momenta by a linear combination of the momenta and the relative coordinate multiplied by an appropriate matrix.

In the other approach, one starts with two *single*-particle Dirac equations [4, 5], adding to each of them interaction terms that are still Poincaré invariant. One then first imposes constraints on these interactions, so that the equations are compatible which implies that in the centre-of-mass frame they are independent of the relative time. There appears then only a single potential V which, again in the centre-of-mass frame, depends only on the magnitude r of the relative coordinate. By adding these two equations, multiplied by appropriate factors, we arrive at a single equation which is identical to the one derived in the previous paragraph.

The energy spectrum of this final equation can be obtained exactly and turns out to give an equally spaced spectrum for E^2, as also happens for the energy E of the non-relativistic oscillator.

Note that in both approaches one can include an extra term $U(\rho)$ with ρ given by equation (17), without changing the Poincaré invariant character of these equations. In the centre-of-mass frame it would imply that the relative position vector $\boldsymbol{r}$ in equation (27) is replaced by $U(r)\boldsymbol{r}$. We then get a very definite problem that can be discussed along the lines leading from equation (27) to (32), but it is unlikely that the latter can be solved in the closed form of the example discussed in this paper.

We will like to extend this conclusion to compare, as suggested by the referees, the results of the present paper with others that have appeared in the literature.

We shall start with our original approach to the relativistic two-body problem with a Dirac-oscillator interaction, described by equation (6) of this paper. From the discussion given in [2], we see that (by a reasoning similar to that of the present paper) the energy spectra of equation (6), associated with the different spins and parities is given by

Spin	Parity	Energy	
0	$(-1)^j$	$E^2 = 4 + 4\omega N,\ E^2 = 0$	(57*a*)
1	$(-1)^j$	$E^2 = 4 + 4\omega(N+1),\ E^2 = 0$	(57*b*)
1	$-(-1)^j$	$P(E^2, N, j) = 0,\ E^2 = 0$	(57*c*)

where the non-zero energies E^2 in (57*c*) were obtained by the solution of a third-degree equation in E^2, so that writing P explicitly [2] we have

$$\{E^2[E^2 - 4 - 4\omega(N+1)][E^2 - 4 - 4\omega(N+2)] - 64\omega^2 j(j+1)\} = 0. \tag{58}$$

Clearly, the energy spectrum of equation (6) given by (57) and that of (27) given by (54) are quite different, except in the case of $s = 0$. In view of the difference between equations (6) and (27) through the scalar term $\gamma_{51}\gamma_{52}$, this should not be more surprising than, for example, the difference between the spectra of the oscillator and Coulomb problems.

Another way to emphasize the difference is to go to the non-relativistic limit of both (6) and (27). For the latter we see from (56) that it corresponds to a Hamiltonian of the form

$$H = \omega(\boldsymbol{\eta} \cdot \boldsymbol{\xi} + \boldsymbol{S}^2) \tag{59}$$

while for the former the non-relativistic limit (which because our units are $\hbar = m = c = 1$ is achieved when $\omega \ll 1$) becomes

$$H' = \omega(\boldsymbol{\eta} \cdot \boldsymbol{\xi} - \boldsymbol{L} \cdot \boldsymbol{S}). \tag{60}$$

Clearly, H *does not* distinguish between the relative orientation of $\boldsymbol{L}$ and $\boldsymbol{S}$ while H' *does*.

There remains an unanswered question of why equation (27) has a degeneracy involving all states, while equation (6) presents this property only for states of parity $(-1)^j$. This is an intriguing question, and the answer to it, following an approach suggested by Quesne

[9] for the one-body Dirac oscillator, seems to stem from the fact that equation (27) has a supersymmetry not present in equation (6). This point of view, which implies considering the solutions of equation (27) both for ω and $-\omega$, will be developed out in a future publication.

Finally, we would like to acknowledge a very important approach to the problem due to Bijtebier [10]. As in the case of Sazdjian [4] and Crater and van Alstine [5], he starts with two relativistic equations restricted by a compatibility condition.

In fact, if p, P are expressed in terms of the original four-momenta of the two particles p_1 and p_2 and V is taken as $V = \tanh\Delta$, with Δ being a function of ρ, equations (23) of the present paper are identical to equations (4.1) of Bijtebier's paper [10]. However, the object of the present paper was not to derive equation (23), which in fact was discussed also in a similar form by Sazdjian [4] and Crater and van Alstine [5], but to establish a relation between the work of all these authors and the approach pursued by Moshinsky *et al* [1, 2]. Furthermore, we discussed a particular form of a Dirac-oscillator interaction which allowed us to solve our problem in an explicit and analytic way. We would also like to acknowledge the pioneering contributions of Bijtebier to aspects of the relativistic many-body problems [11].

A question was also raised about the possible applications of the problem discussed in this article. While the present analysis had the purely conceptual objective of relating two very different approaches to the relativistic and quantum mechanical two-body problem, the general viewpoint followed by Moshinsky *et al* [12, 13] for particle–antiparticle systems as well as for three body problems has had applications to meson and baryon mass spectra [13, 14].

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